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Guest Editorial

Antoni Zygmund and His Work

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A. INTRODUCTION

Antoni Zygmund was born in Warsaw, Poland, on December 26, 1900. He died on May 30, 1992, in Chicago. Thus ended a very successful life, rich in mathematical discoveries, wonderful students, and exemplary books.

The largest part of Zygmund's work was devoted to one subject: the theory of trigonometric series. This beautiful theory became possible after the discovery of the Lebesgue integral, and it achieved its highest ambition with Carleson's theorem in 1966. Zygmund was one of its builders, together with Hardy, Littlewood, Kolmogorov, Lusin, and others. At present, this theory is part of harmonic analysis, together with topics such as Banach algebras and harmonic analysis on groups.

Is Fourier analysis part of approximation theory? I think it is better considered as an allied theory that often appears as a tool, similar in this respect to orthogonal polynomials and to wavelets. The readers of this journal may ask, what part of Zygmund's work is most relevant to approximation theory? If we put aside Zygmund's theory of singular integrals, then I believe the answer is simple. It is those of his achievements which are most important to all general analysts.

I begin my review with these simple, important results. Then I discuss, as examples, a few of his many special investigations. I end with a description of my personal encounter with Zygmund's work.

B. ZYGMUND'S RESULTS WHICH EVERY APPROXIMATION THEORIST SHOULD KNOW

B.1. *Bernstein's inequality* for trigonometric polynomials T_n of degree at most n on the circle $\mathbb{T}$ is $\|T_n'\|_\infty \leq n \|T_n\|_\infty$. Since approximation theory is

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most often used in L_p spaces, one constantly applies Zygmund's inequality [3],

$$\|T_n'\|_p \leqslant n \|T_n\|_p, \qquad 1 \leqslant p < \infty. \tag{1}$$

Later extensions of this include the pointwise relation $T_n'(x) \prec nT_x(x)$ on $\mathbb{T}$ (with the quasi-inequality $\prec$ of Hardy and Littlewood), so that (1) is true for any rearrangement-invariant Banach function space in place of L_p, and the fact due to Arestov that (1) is valid also for the non-Banach spaces L_p with $0 < p < 1$.

B.2. *Function spaces.* Zygmund was one of the first to study the following spaces:

(1) The space $L \log L$, which consists of all functions f on $\mathbb{T}$ that satisfy

$$|f|_{L \log L} \overset{\text{def}}{=} \int_{\mathbb{T}} |f(x)| \log^+ |f(x)|\, dx < +\infty.$$

He proved [2] that for $f \in L \log L$, the conjugate function $\tilde{f}(x)$ (which exists a.e. if $f \in L_1$ but is not necessarily integrable) belongs to the space L_1 and satisfies $\|\tilde{f}\|_1 \leqslant C\, |f|_{L \log L}$, where C is a constant.

(2) The space V_α, $0 < \alpha < 1$, which consists of all functions of bounded variation on $\mathbb{T}$ that also belong to Lip α. Zygmund [1] proved that the Fourier series of functions $f \in V_\alpha$ are absolutely convergent.

These spaces continue to appear in modern approximation theory. For example, let us compare the error of approximation $\rho_n(f)$ of f by rational functions of degree at most n with the polynomial approximation error $E_n(f)$. For typical functions f of Sobolev or Lipschitz spaces, they are of the same order. This is not true for the spaces given above. For example, $\rho_n(f) \leqslant C(\log n/n)$ if $f \in V_\alpha$, $0 < \alpha < 1$ (Petrushev, Pekarskii), while only $E_n(f) \leqslant Cn^{-\alpha}$ is true. Similarly, one has $\rho_n(f) \leqslant C(1/n)\, |f'|_{L \log L}$ if $f' \in L \log L$ (Pekarskii), but only $E_n(f) \leqslant C(1/n)\, \|f'\|_{L_\infty}$ holds for polynomial approximation.

B.3. *Smooth functions.* Since 1912 it has been known that the error of the trigonometric polynomial approximation $E_n(f) =^{\text{def}} \min_{T_n} \|f - T_n\|_\infty$ on the circle $\mathbb{T}$ satisfies $E_n(f) = O(n^{-\alpha})$, $0 < \alpha < 1$ if and only if $f \in \operatorname{Lip} \alpha$. For $\alpha = 1$ this is not true. Although $f \in \operatorname{Lip} 1$ implies that $E_n(f) = O(n^{-1})$, the converse no longer holds. This phenomenon was explained by Zygmund [5] in 1945. The relation $E_n(f) = O(n^{-1})$ is equivalent not to $\Delta_h f(x) =^{\text{def}} f(x+h) - f(x) = O(h)$, but to the weaker assumption

$\Delta_h^2 f(x) =^{\text{def}} f(x+2h) - 2f(x+h) + f(x) = O(h)$. The situation repeats itself for $E_n(f) = O(n^{-r})$, with integer $r > 0$. This has led to two contrasting definitions of the Lipschitz spaces Lip α, $\alpha > 0$. We define $\omega_k(f, h) =^{\text{def}} \max_{|t| \leqslant h} |\Delta_t^k f(x)|$, $k = 1, 2, \ldots$; this is the kth modulus of smoothness of f. If $\alpha = r + \beta$, $0 < \beta \leqslant 1$, and $r = 0, 1, \ldots$, then $f \in \text{Lip}\,\alpha$ means that $f^{(r)} \in \text{Lip}\,\beta$ or, equivalently, that $\omega_1(f^{(r)}, h) = O(h^\beta)$. On the other hand, with $r^* = [\alpha] + 1$, $f \in \text{Lip}^*\,\alpha$ means that $\omega_{r^*}(f, h) = O(h^\alpha)$.

The two Lipschitz spaces are identical if α is not an integer. For $\alpha = 1, 2, \ldots$, $f \in \text{Lip}^*\,\alpha \Leftrightarrow E_n(f) = O(n^{-\alpha})$. This space is separable, while Lip α, $\alpha = 1, 2, \ldots$ is not. Nevertheless, the spaces Lip α and Lip* α are similar for $\alpha = 1, 2, \ldots$. Their unit balls have the same entropy in the uniform metric.

B.4. *Interpolation of operators.* The Riesz–Thorin interpolation theorem and operators of strong or weak type (p, q), $1 \leqslant p \leqslant q \leqslant \infty$, are well known. In [7], Zygmund reconstructs and generalizes the proof of a theorem of Marcinkiewicz: if a linear operator U is of weak types (p_j, q_j) with constants M_j, $j = 1, 2$, then for each $0 < \theta < 1$, U is of strong type (p, q), $1/p = \theta/p_1 + (1-\theta)/p_2$, $1/q = \theta/q_1 + (1-\theta)/q_2$, with norm $\leqslant (C/\theta(1-\theta)) \max(M_1, M_2)$ (today, one usually formulates this theorem in terms of the $L_{p,r}$-spaces). This paper of Zygmund led to a flowering of the theory of interpolation of operators in the sixties and seventies (another source was the K-functionals and J-functionals of Peetre).

A simple variation of this theorem is that if U belongs to the strong types $(1, 1)$ and (∞, ∞), then it maps any rearrangement invariant Banach function space (that is, "any space between L_1 and L_∞") onto itself with norm less than or equal to $C \max(M_1, M_2)$. For sequences instead of functions, this is a theorem of Hardy, Littlewood, and Pólya already obtained in 1929 with a similar proof. For functions, it is a folk theorem, used in different forms by many authors.

C. A Selection of More Special Achievements by Zygmund

All of this is quite important, but does not necessarily represent the deepest, or the most difficult of Zygmund's achievements. Nevertheless, Zygmund was a powerful technician. The subjects he liked most were summability of trigonometric series, conjugate Fourier series and functions, trigonometric and power series with random coefficients such as $\sum \pm c_j e^{ijx}$, lacunary series, differentiability properties of functions, and, as the crowning achievement, his and Calderón's theory of singular integrals.

C.1. *Differentiation of functions.* In [4], Zygmund addresses the following question. How can one define the kth derivative, $k>1$, of a function of one variable $f(x)$ at x_0, if f is given around x_0, and if one does not know the values or even the existence of lower derivatives? The kth Riemann derivative $D_k f(x_0)$ is the limit (if it exists) $\lim_{h\to 0+} h^{-k} \bar{\Delta}_h^k(f, x_0)$ where $\bar{\Delta}_h^k$ is the symmetric difference at x_0; the Peano derivative $D^k f(x_0)$ is the number a_k in the expansion (if it exists) $f(x_0+h)=a_0+a_1h+\cdots+(a_k/k!)h^k+o(h^k)$. Clearly, $D_k f(x_0)$ exists whenever the latter holds. The converse is by no means obvious, but is true almost everywhere.

C.2. *Sets of uniqueness.* A measurable set $A\subset\mathbb{T}$ is a *set of uniqueness* if each trigonometric series that converges to zero outside A is identically zero. Each set of uniqueness has measure zero. The problem of characterizing all sets of uniqueness seems to be hopeless; one can only study examples. After preliminary work by Salem and Piatetski-Shapiro the following astonishing theorem has finally been proved [6]. We construct perfect sets in the following standard way. Let $0<\theta<1$ be fixed. From $[0, 2\pi]$ we omit a central closed interval of length $\theta\cdot 2\pi$, in each of the remaining intervals I_1, I_2 we again omit the central interval of length $\theta\,|I_1|=\theta\,|I_2|$, and so on. The remaining perfect set of measure zero is a unicity set if and only if $\theta=1/\xi$, where ξ is a Pisot number. This means that ξ $(\xi>1)$ is a real root of a monic polynomial P with integral coefficients, while all other roots of P are complex numbers less than one in absolute value.

C.3. *The Littlewood–Paley theory,* as created before the war, had as its first objective the proof of

$$s_{2^k}(x)\to f(x) \qquad \text{a.e.} \tag{2}$$

for the Fourier sums $s_n(x)$ of functions $f\in L_p$, $1<p<\infty$ (the case $p=2$ is easy). With the participation of Zygmund and Marcinkiewicz, and with the help of complex analysis, a powerful technique was developed. Let $u(r\cos t, r\sin t) =^{\text{def}} u(x, y)$ be the Poisson integral of f. The theory uses the Littlewood–Paley function

$$g_f(t) \overset{\text{def}}{=} \left(\int_0^1 (1-r)\,|\nabla u|^2\,dr\right)^{1/2}, \qquad t\in\mathbb{T},$$

with $|\nabla u|^2=(\partial u/\partial x)^2+(\partial u/\partial y)^2$, and the Lusin function

$$s_f(t) \overset{\text{def}}{=} \left(\int_{\Omega_\delta(t)} |\nabla u|^2\,dx\,dy\right)^{1/2}, \qquad 0<\delta<1,$$

where $\Omega_\delta(t)$ is the angular region consisting of the disk $|z| \leqslant \delta$ and the area between it and its two tangents, emanating from e^{it}. One has $C_1 \|f\|_p \leqslant \|g_f\|_p \leqslant C_2 \|f\|_p$ and $C_1 \|f\|_p \leqslant \|s_f\|_p \leqslant C_2 \|f\|_p$ for $f \in L_p$, $1 < p < \infty$. Applications are to the existence of non-tangential limits of analytic functions, to multipliers, and to estimation of the function

$$\gamma_f(t) \stackrel{\text{def}}{=} \left(\sum_{n=0}^{\infty} \frac{1}{n+1} (s_n(x) - \sigma_n(x))^2 \right)^{1/2},$$

where σ_n is the nth Fourier mean of f. The latter leads to a proof of (2). It is true that (2) became obsolete in 1967 when Hunt proved that convergence takes place for the entire sequence $s_n(x)$. This, however, did not destroy the importance of the theory, especially when it was shown (see, e.g., Stein [10]) that the theory applies to functions of n variables, if one replaces complex analysis by the theory of singular integrals or by harmonic functions of n variables.

C.4. *Singular integrals of Zygmund and Calderón* [8]. This deep and important theory stands somewhat apart from the rest of Zygmund's work. Achieved late in his life, it is an n-dimensional real variable theory, with new techniques. With its applications to differential equations, it influences even applied mathematics (cf. [9]).

For $n = 1$, the Hilbert transform of a function $f \in L_1(\mathbb{R})$ is given by the formula

$$\tilde{f}(x) = \lim_{\varepsilon \to 0} \int_{|x-y| \geqslant \varepsilon} \frac{f(y)}{x-y}\, dy$$

(this is closely related to the conjugate of f which is also often denoted by $\tilde{f}$). For points $x = (x_1, ..., x_n)$ of $\mathbb{R}^n$ we put $|x| =^{\text{def}} (\sum x_i^2)^{1/2}$ and define the unit sphere Σ by $|x| = 1$. A singular integral of $f \in L_1(\mathbb{R}^n)$ is given by a kernel $K(x) = \Omega(x')\,|x|^{-n}$, $x' =^{\text{def}} x/|x| \in \Sigma$, where $\int_\Sigma \Omega(x')\, dx' = 0$, and the formula

$$\tilde{f}(x) = \lim_{\varepsilon \to \infty} \int_{|x-y| \geqslant \varepsilon} f(y)\, K(x, y)\, dy. \tag{3}$$

The main theorem is that for smooth Ω and all $f \in L_p(\mathbb{R}^n)$, $1 < p < \infty$, $\tilde{f}(x)$ exists a.e. and is a bounded map of L_p into itself. Generalizations of (3) include kernels $K(x, x-y)$. This is the bare minimum of the theory that we can give here. Applications include an extension of the Littlewood–Paley theory to n-dimensions, to theorems about the differentiability of functions (Zygmund with Stein), and to theorems about the existence and smoothness properties of solutions of partial differential equations (some first important papers were by Zygmund and Calderón). The reader may consult the book of Stein [10] and, for newer developments, the one by Christ [11].

D. Zygmund's Books

Zygmund wrote two books on trigonometric series, each of them a perfect exposition of this theory at the time of its publication. The first book appeared in 1935 as Volume 5 of *Monografje Matematiczne*, Warsaw, 332 pages (the first volume of this series was Banach's "Théorie des Opérations Linéaires"). His second book was the 1959 publication by Cambridge University Press in 2 volumes, 737 total pages. J. P. Kahane calls this book "The Bible" of a harmonic analyst. (With Saks, Zygmund also published in 1938 a successful book on the theory of analytic functions.)

The story of my very personal encounter with Zygmund's first book, which was translated into Russian in 1939, begins in early May 1942. During the height of World War II, I ended up in Kislovodsk in the Caucasus, evacuated from Leningrad with the Pedagogical Institute Herzen. This was after several years of economic depression in Leningrad and after the horrible winter of 1941–1942 during the German blockade. I was determined to start my mathematical life anew. In the nearby city of Pyatigorsk, in the downtown library of a college, next to a proud six-story building, I found and borrowed Zygmund's book. In a few months the Germans arrived. The six-story building burned down (it turned out to be the headquarters of the KGB, the secret police of the region), and with it went the humble library. I thus could not return Zygmund's book. I continued to study it in Poland, where I moved with my wife and new-born son in 1943. We had to migrate as far as possible westward. Somehow I managed to write a few mathematical papers, one of them on Fourier series. I sent them to Professor K. Knopp for the *Mathematische Zeitschrift*. As a result, in 1944 we ended up in Tübingen, in the western part of Germany, where I worked with Professors Kamke and Knopp until the beginning of 1949. I was still clinging to Zygmund's book. My so-called "Lorentz spaces" were an outcome of this study. In addition, my first Ph.D. student, W. B. Jurkat, wrote his dissertation on Fourier series in 1949. This all proves, I hope, that Zygmund's book was, at the time, an excellent introduction to classical analysis.

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